BIMA
Technical Memo #76

On the Use of Shifted m-sequences for Phase Switching
Or
Why not Complex Walsh Functions?

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Abstract

Recently Eric Keto of the Smithsonian Astrophysical Observatory (Technical Memo Number 134) suggested using four level, shifted m-sequences as an alternative to two level Walsh functions used for phase switching in interferometer arrays. Phase switching is used to reduce DC level shifts, to separate the sidebands, and it is used to provide cross-talk and interference rejection. In order for cross-talk and interference rejection to work, it is necessary for all of the phase demodulation functions for each of the baselines to be orthogonal. It is also necessary for all of the antenna phase modulators to be orthogonal. As bigger arrays are designed and the need continues for shorter integrations, there is a need for more efficient switching schemes. The requirement for baseline and antenna orthogonality places severe constraints on the selection of switching functions. The crucial question that must be answered in judging any modulation scheme is how efficient it is in supporting the largest antenna array in the fewest steps. We examine m-sequences, normal two level Walsh functions, and two kinds of complex Walsh functions. It appears that, for large systems, even an ideal solution may produce a set of phase modulation functions that are too long for short integrations.

Discussion

Phase switching in interferometry refers to a method whereby the first and/or second local oscillator in each of the antenna receivers is modulated in order to remove DC drift, separate the sidebands and eliminate cross-talk between the antennas. When the local oscillators of two antennas are in phase, their cross-correlation will be positive. If they are 180 degrees out of phase relative to each other then their cross-correlation will be negative. The difference between these two measurements eliminates all instrumental DC and provides a positive real cross-correlation measurement. If the two antennas are 90 degrees out of phase with each other then a positive imaginary cross-correlation measurement will occur. If they are 270 degrees out of phase then the result will be a negative imaginary cross-correlation measurement. Once again, the difference between these measurements provides a positive imaginary cross-correlation measurement. Sideband separation occurs when a Fourier transform is taken of the real and imaginary cross-correlation functions taken together as a complex cross-correlation. Cross-talk may be eliminated by using switching functions that are orthogonal. Square waves could be used with a period that increases by a factor of two at each antenna. As differences are taken interference from the wrong antenna will be averaged to zero. Square waves are not a very efficient way of doing it since adding a new antenna requires a factor of two more time to complete the sequence. Walsh functions are much more compact since 256 different orthogonal functions occupy only 256 steps. Unfortunately, Walsh functions have only two states, +1 and -1. Eric Keto has suggested a set of orthogonal functions that have four states.

The Walsh switching schemes used at the Hat Creek Radio Observatory and at the Owens Valley Radio Observatory are functionally almost identical. At Hat Creek and Owens Valley, in order to separate the sidebands, the first LO is modulated by 90 degrees using a Walsh function. The first LO is also switched by 180 degrees using a member of another Walsh set that is completed during the time required for each of the steps in the 90 degree sequence. At Hat Creek 180 degree switching is accomplished at a second LO using another member of the 90 degree Walsh set. It is important to recognize that, at Hat Creek, 180 degree switching could have been accomplished by modulating the first LO. The first LO would be modulated by the simple sum of 180 and 90 degrees depending upon the states of the two members of the same Walsh set and this could be done using exactly the same functions that are being used today. At Hat Creek a set of 256 Walsh functions requiring 256 steps to complete is being used. At Owens Valley a set of 8 Walsh functions is used to modulate by 90 degrees and a faster set of 8 Walsh functions is used to modulate by 180 degrees for a total of 64 steps. The Owens Valley system isolates the two functions of DC removal and sideband separation. The main difference in the two schemes is the way they are demodulated.

In order to see that the baseline demodulation functions must be orthogonal, it is only necessary to assume that two baselines in a three-antenna system have exactly the same demodulation function. If there were cross-talk between two of the antennas (say A and B) then baselines AC and BC would both demodulate the cross-talk if they used the same demodulation function. If AC and BC used orthogonal demodulation functions then the cross-talk would average to zero. By the same token, if antennas A and B were both modulated by the same function then there would be no way to distinguish between A and B if cross-talk occurred. It is tempting to assume that orthogonal modulation functions at the antennas will produce demodulation functions at the baselines that are all orthogonal, however this is not the case. In an ideal situation, an array of 16 antennas could theoretically be modulated with 16 orthogonal 16-step sequences. A 16 dimensional space has no more than 16 orthogonal sequences. Demodulation at the baselines must be done with a 16-step sequence. A 16-antenna array has 120 baselines. In order to provide 120 orthogonal demodulation functions it is necessary to have, at an absolute minimum, 120 steps.

It has been suggested to me that the requirements for orthogonality could be relaxed in the case where baselines AB and CD are involved. If baselines AB and CD both have the same demodulation function then cross-talk would have to occur between both A and C and B and D. While this situation may be considered to be less likely it can still occur. Unfortunately the flaw that allows cross-talk to occur between A and C could be the same flaw that allows cross-talk to occur between B and D. A few years ago I was sent to Hat Creek to repair a "design flaw" in a telemetry system I designed. At that time all of the signals to and from each antenna in the BIMA array went through a single co-ax. All of the co-axes entered the main building through feed-through connectors mounted on an aluminum plate. That plate conspired with those connectors to make a large adding network that mixed signals from all of the antennas. Even though there was sufficient cross-talk to cause occasional errors in the telemetry, the system continued to produce flawless images.

Walsh Functions

Things become quite complex when Walsh functions are used to modulate both the 90 degree and the 180 degree phase switching. Jack Welch and Eric Keto have both derived the demodulation functions required for two state orthogonal functions. I will outline here briefly the conditions required for baseline orthogonality.

Given Walsh functions W0j W0k W1j W1k for antennas j & k where W0 modulates by 90 degrees and W1 modulates by 180 degrees. The baseline demodulation functions are

```
     Real C(W)jk = W1j*W1k*(1+W0j*W0k)
     Imag S(W)jk = W1j*W1k*W0j*(1-W0j*W0k)
```

where

```
     W0j*W0k          switches between real and imaginary
     W1j*W1k          determines the sign of the real
     W1j*W1k*W0j      determines the sign of the imaginary
```

It is necessary that

```
     C(W)jk dot C(W)mn = 0
```

and

```
     S(W)jk dot S(W)mn = 0
```

for all jk != mn.

(* represents the vector product of the Walsh functions and dot represents the scalar product.)

Considering C(W)

```
     [W1j*W1k*(1+W0j*W0k)]dot[W1m*W1n*(1+W0m*W0n)]=0
     [W1j*W1k+W1j*W1k*W0j*W0k]dot[W1m*W1n+W1m*W1n*W0m*W0n]=0
     [W1j*W1k]dot[W1m*W1n]+
     [W1j*W1k]dot[W1m*W1n*W0m*W0n]+
     [W1j*W1k*W0j*W0k]dot[W1m*W1n]+
     [W1j*W1k*W0j*W0k]dot[W1m*W1n*W0m*W0n] = 0
```

or since the product of two Walsh functions is another Walsh function and all Walsh functions are orthogonal.

```
     (1) [W1j*W1k] != [W1m*W1n]
     (2) [W1j*W1k] != [W1m*W1n*W0m*W0n]
     (3) [W1j*W1k*W0j*W0k] != [W1m*W1n]
     (4) [W1j*W1k*W0j*W0k] != [W1m*W1n*W0m*W0n]
```

The Walsh products W1a*W1b and W1a*W1b*W0a*W0b must all be mutually exclusive.

Considering S(W) a similar result

```
     (1) [W1j*W1k*W0j] != [W1m*W1n*W0m]
     (2) [W1j*W1k*W0j] != [W1m*W1n*W0m*W0n]
     (3) [W1j*W1k*W0j*W0k] != [W1m*W1n*W0m]
     (4) [W1j*W1k*W0j*W0k] != [W1m*W1n*W0m*W0n]
```

The Walsh products W1a*W1b*W0a and W1a*W1b*W0a*W0b must all be mutually exclusive.

While 256 Walsh functions may seem like more than enough to satisfy the requirements of a 45 baseline 10 antenna interferometer like the BIMA array at Hat Creek it, just barely makes it. We have constructed a computer program to find a set of valid Walsh functions for an interferometer of any size. If one more antenna is added to our system, the above conditions for orthogonality cannot be met. If the Walsh sequence is doubled to 512 separate Walsh functions then a solution for only 13 antennas or 78 baselines can be found. If the Walsh sequence is extended to 1024 steps then a solution exists for 17 antennas or 136 baselines.

Four State Sequences

Demodulation is rather simple with four state sequences. If A represents a sequence modulating one antenna and B a sequence modulating another then the cross-correlator output from these two antennas will be the complex vector product of the two sequences.

$$M = A \cdot B^*$$

Any time M is in the +1 state the cross-correlation function is positive real. If it is -1 it will be negative real. If it is +j then it will be positive imaginary, and if it is -j then the cross-correlation function will be negative imaginary.

If the real part of M is identical to the real part of another demodulation function then it is possible for cross-talk to occur in the real part of the cross-correlation measurement. In order to assure that this does not happen the four level functions should be constrained in the same way that the two level functions are. Given baseline demodulation functions M and N:

$$2 \cdot \text{Real } M = M + M^*$$
$$2 \cdot \text{Imag } M = M - M^*$$

It is necessary that

$$\text{Real } M \text{ dot Real } N = 0$$
$$\text{Imag } M \text{ dot Imag } N = 0$$

for all different baselines.

Considering the real part:

$$(M+M^*) \text{ dot } (N+N^*) = 0$$
$$M\text{dot}N+M\text{dot}N^*+M^*\text{dot}N+M^*\text{dot}N^* = 0$$

A similar result for the imaginary part implies that not only must a demodulation function be orthogonal to all other demodulation functions but its complex conjugate must be as well. Similar reasoning applies to the modulation functions used at the antennas.

In order for demodulation to work, the correlator must spend the same amount of time in each state. Any switching function used as a demodulator must, therefore, have the same number of steps in each of the four states. A similar constraint is not required for the antennas since, conceivably, one antenna may not switch at all while the antenna it's being correlated with does all of the switching. The only constraints on the antenna modulators are that they be orthogonal and not the complex conjugates of each other.

Requirements for Demodulation

Given a potential baseline set of demodulator sequences M_n and any antenna modulator sequence A_0 a set of orthogonal antenna modulators may be found by simple multiplication. If, for example, A_0 and A_1 are two sequences feeding a baseline, the baseline may be demodulated by a sequence M_0 where

$$A_1 \cdot A_0^* = M_0$$

We can solve for A_1 given A_0 and M_0;

$$A_1 \cdot A_0^* \cdot A_0 = M_0 \cdot A_0$$

$$A_1 = M_0 \cdot A_0$$

All of the antennas in the array will be cross correlated with A_0 so that given any known demodulation sequence M_n the other antenna sequences may be derived.

$$A_2 = M_1 \cdot A_0$$

$$A_3 = M_2 \cdot A_0$$

....

$$A_m = M_{m-1} \cdot A_0$$

Since all of the antenna sequences are now fixed given an initial set of baseline demodulators, all of the other baseline demodulators can be derived. Given any two antennas A_2 and A_3, the baseline demodulator is

$$A_2 \cdot A_3^* = M_1 \cdot A_0 \cdot M_2^* \cdot A_0^* = M_1 \cdot M_2^*$$

					A_0
				A_1	M_0
			A_2	$M_0 \cdot M_1^*$	M_1
		A_3	$M_1 \cdot M_2^*$	$M_0 \cdot M_2^*$	M_2
	…	…	…	…	…
A_{n+1}	…	$M_2 \cdot M_n^*$	$M_1 \cdot M_n^*$	$M_0 \cdot M_n^*$	M_n

Table of Antennas and Their Baseline Demodulators

The baseline demodulators can all be selected independent of any antenna modulators. Further, any set of antenna sequences selected in the above manner will be orthogonal since

$$A_2 \text{dot} A_3 = \Sigma\ A_2 \cdot A_3^* = \Sigma\ M_1 \cdot M_2^* = 0$$

because all of the M_n are orthogonal. Considering the special case of A_0

$$A_0 \text{dot} A_1 = \Sigma\ A_0 \cdot A_1^* = \Sigma\ M_0^* = 0$$

because M_0 has an equal number of steps of 1, -1, j and -j.

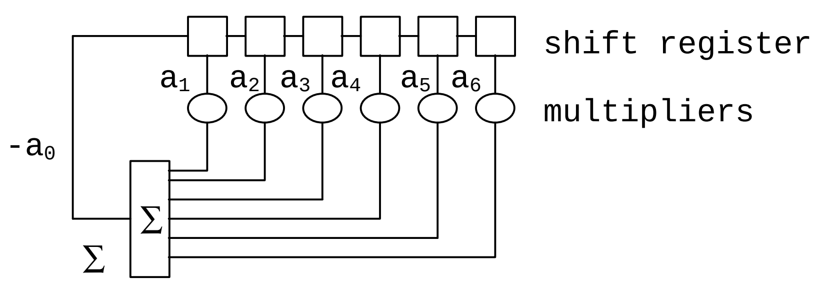
The above observations may be used to build a general purpose demodulation selection program to be used for evaluating any set of orthogonal sequences. The selection program need not consider the antenna sequences at all but need only concentrate on finding a suitable set of baseline demodulators.

Shifted m-sequences

M-sequences or maximal length pseudo-random sequences are probably more familiar to people in the binary form. These sequences have a length equal to

$$\text{len} = q^m - 1$$

where len is the total length of the sequence, q is the number of levels in each element of the sequence and m is the length of the shift-register used to generate the sequence. The circuit appears as follows



The shift register is shifting from left to right and each element of the shift register must be wide enough to represent all of the levels. If q = 3 then the width must be two bits. If each element of the shift register is viewed as a delay operator then the above loop can be expressed as a polynomial where Δ represents a delay.

$$0 = a_0 + \Delta a_1 + \Delta^2 a_2 + \Delta^3 a_3 + \Delta^4 a_4 + \Delta^5 a_5 + \Delta^6 a_6$$

The arithmetic used for this polynomial is to the modulus of the number of levels. Expressed in the C programming language, c = (a+b)%q and c = (a*b)%q

When this circuit is turned on, it produces a stream of numbers that look very much like white noise. It has an auto-correlation function that is an impulse at zero lag and all the rest of the lags are a very small constant number until it reaches a lag that corresponds to the end of the sequence where it repeats the impulse and sequence of almost zero lags. It is a completely predictable noise source.

This very mysterious circuit will work for all polynomials that cannot be factored or expressed as the ratio of two polynomials (primitive and irreducible). They are called maximal length because they go through all possible combinations of q things taken m at a time. It is easily seen why one of the combinations is missing. If all zeros enter the system, it grinds to a halt. Unfortunately this circuit, as described, works only with q equal to a prime number. If you want a system that works with 4 levels you must change the rules slightly.

+	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

Addition

.	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

Multiplication

With modified arithmetic the same principals apply and a set of maximal length four level sequences becomes available. Unfortunately the auto-correlation of these sequences is not quite as clean as in the case of prime level modulo arithmetic. The expected spike appears at zero lag but smaller spikes also appear at 1/3 and 2/3 the total length of the sequence.

Neither prime level nor four level maximal length sequences are exactly orthogonal. Eric Keto provides two methods to make the sequences orthogonal. First, the missing state must be added to the beginning of each of the sequences. In the case of prime level sequences, this modification is all that is necessary. There will be q^m-1 orthogonal sequences of length q^m. The case of four level sequences is a little more difficult. Adding the missing state to the beginning of each sequence results in only the first third of the sequences being orthogonal. In order to make the full set of q^m-1 shifted sequences orthogonal, you change the signs of either the real or the imaginary numbers and concatenate it with the original sequence extending the total number of steps required to complete the sequence to $2*q^m$.

Shifted m-sequences as modified by Eric Keto have two characteristics which make them promising as demodulation functions. First, they are an orthogonal set. Second, they give equal representation to the products 1, -1, j and -j over the sequence interval. The vector product of any two Keto sequences is not a member of the original set. The interesting thing is the fact that all of the vector products appear to have equal numbers of each of the states.

Complex Walsh Functions

The following scheme originated with Ferdinand R. Ohnsorg of the Systems and Research Division Honeywell Inc. in an article that appeared in the "Applications of Walsh Functions" 1970 proceedings symposium and workshop held at the Naval Research Laboratory, Washington, D. C.

A set of complex Walsh functions that take on the values of +1, -1, +j and -j may be generated in a manner similar to a method used for ordinary Walsh functions. A Hadamard array is generated. For ordinary Walsh functions each element of the Hadamard array is defined as

$$H_{mn} = (-1)^{m.n}$$

where m represents the column index and n the row index. The product m.n is the scalar product of the binary representation of m and n treated as vectors.

m

	0	1	2	3	4	5	6	7
0	+1	+1	+1	+1	+1	+1	+1	+1
1	+1	-1	+1	-1	+1	-1	+1	-1
2	+1	+1	-1	-1	+1	+1	-1	-1
3	+1	-1	-1	+1	+1	-1	-1	+1
4	+1	+1	+1	+1	-1	-1	-1	-1
5	+1	-1	+1	-1	-1	+1	-1	+1
6	+1	+1	-1	-1	-1	-1	+1	+1
7	+1	-1	-1	+1	-1	+1	+1	-1

Hadamard Array

Each of the rows of the above array represents a different Walsh function.

A modified complex array results if we change the ritual.

$$M_{mn} = (-1)^{m.n}(-j)^{m.n/2}$$

	0	1	2	3	4	5	6	7
0	+1	+1	+1	+1	+1	+1	+1	+1
1	+1	-1	+1	-1	+1	-1	+1	-1
2	+1	-j	-1	+j	+1	-j	-1	+j
3	+1	+j	-1	-j	+1	+j	-1	-j
4	+1	+1	-j	-j	-1	-1	+j	+j
5	+1	-1	-j	+j	-1	+1	+j	-j
6	+1	+1	+j	+j	-1	-1	-j	-j
7	+1	-1	+j	-j	-1	+1	-j	+j

Modified Array

Each row of the above modified array is a suitable complex Walsh function. The length of these Walsh functions is a power of 2. Each of the terms 1, -1, j and -j appear an equal number of times except for the first two sequences making all but two of them suitable for demodulation. Unlike the m-sequences some of the vector products in this set do belong to the original set but not all of them.

Yet More Complex Walsh Functions

Another approach to complex Walsh function formation is provided by H.E. Chrestenson. This approach relies on the use of multilevel Rademacher functions. Two level Rademacher functions are simple square waves. The lowest (zero) order Rademacher function is a square wave that starts as a 1 and ends as a -1. The next function (one) is twice the frequency also starting with a 1. The functions increase in order with each factor of two increase in frequency. Two level Walsh functions are generated from these Rademacher functions by first expressing the Walsh sequency as a binary number. The least significant bit of this number is used as an exponent for the zero order Rademacher function. The next bit is used as the exponent for the next order function until all bits are used. All of these functions are multiplied together to produce the Walsh function of the desired sequency.

$$Wal_n = (Rad_0)^{v0}(Rad_1)^{v1}(Rad_2)^{v2}(Rad_3)^{v3}\ldots$$

Where

$$n = v0+2*v1+4*v2+8*v3\ldots \quad \text{all } v = 0 \text{ or } 1$$

In order to produce complex Walsh functions, the lowest order (zero) four level Rademacher function must take on the following values in sequence, 1, j, -1, -j. The next order (one) is 4 times the frequency of the lowest order. It goes through a complete cycle of the same sequence for each state of the zero order Rademacher function. This time, Walsh sequency must be expressed as a four state number rather than a two state binary number. Each digit in the 4 state sequency number acts as an exponent for the corresponding 4 level Rademacher function.

$$Wal_n = (Rad_0)^{v0}(Rad_1)^{v1}(Rad_2)^{v2}(Rad_3)^{v3}\dots$$

Where

$$n = v0+4*v1+16*v2+64*v3\dots \quad \text{all } v = 0,\ 1,\ 2 \text{ or } 3$$

+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1
+1	+1	+1	+1	+j	+j	+j	+j	-1	-1	-1	-1	-j	-j	-j	-j
+1	+1	+1	+1	-1	-1	-1	-1	+1	+1	+1	+1	-1	-1	-1	-1
+1	+1	+1	+1	-j	-j	-j	-j	-1	-1	-1	-1	+j	+j	+j	+j
+1	+j	-1	-j	+1	+j	-1	-j	+1	+j	-1	-j	+1	+j	-1	-j
+1	+j	-1	-j	+j	-1	-j	+1	-1	-j	+1	+j	-j	+1	+j	-1
+1	+j	-1	-j	-1	-j	+1	+j	+1	+j	-1	-j	-1	-j	+1	+j
+1	+j	-1	-j	-j	+1	+j	-1	-1	-j	+1	+j	+j	-1	-j	+1
+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1
+1	-1	+1	-1	+j	-j	+j	-j	-1	+1	-1	+1	-j	+j	-j	+j
+1	-1	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1
+1	-1	+1	-1	-j	+j	-j	+j	-1	+1	-1	+1	+j	-j	+j	-j
+1	-j	-1	+j	+1	-j	-1	+j	+1	-j	-1	+j	+1	-j	-1	+j
+1	-j	-1	+j	+j	+1	-j	-1	-1	+j	+1	-j	-j	-1	+j	+1
+1	-j	-1	+j	-1	+j	+1	-j	+1	-j	-1	+j	-1	+j	+1	-j
+1	-j	-1	+j	-j	-1	+j	+1	-1	+j	+1	-j	+j	+1	-j	-1

Rademacher derived Walsh functions in sequency order.

Unlike the Hadamard derived Walsh functions, which must be a power of two in length, the Rademacher derived Walsh functions must be a power of four in length. Not all of the Rademacher Walsh functions have an equal number of 1, j, -1, and -j states so more Hadamard Walsh functions make suitable demodulation functions.

The Rademacher derived Walsh functions have a characteristic that is similar to two level Walsh functions. The complex vector product of any two functions results in another function that is a member of the original set. The sequency of the complex vector product of any two functions is the mod 4 sum of the sequencies of the two functions. The sequency is expressed as a four state number having values 0, 1, 2, and 3 for each of the digits. Each of the digits of the two numbers are added without carry. The resulting number is the sequency of the product. This characteristic makes finding baseline demodulators much easier than the trial and error methods required for the other sets.

The Results of Testing

A program was written to find the maximum number of antennas that could be supported by a given set of orthogonal functions. M-sequencies (m-seq.), Hadamard derived (H-Wal) and Radamacher derived (R-Wal) Walsh functions were tested. A second program was written to test only the Radamacher derived functions since they are more predictable and larger arrays can be handled more efficiently. Both programs produced the same results on the smaller arrays. The test results are compared to our current way of modulating the system (Wal) at Hat Creek.

Steps	Wal	m-seq.	H-Wal	R-Wal
16	3	X	4	4
32	4	3	6	X
64	6	X	7	8
128	7	5	10	X
256	10	X	12	15
512	13		15	X
1024	17			22

Antennas supported for a given number of steps.

The clear winner is the set of Radamacher derived Walsh functions. Radamacher derived Walsh functions, however, increase in steps by factor of four jumps. Hadamard derived Walsh functions may have an advantage in situations where a factor of two jump is adequate for the needs of the array. The Hat Creek array at ten antennas could, for instance, reduce its current 256 step switching sequence to 128 steps and reduce its minimum integration time by a factor of two by using a set of Hadamard derived Walsh functions, whereas a Radamacher derived set would require 256 steps.

Conclusions

Even the most efficient set of functions is remarkably wasteful. At 1024 steps the Radamacher derived functions can take care of only 22 antennas or 231 baselines. Out of 1024 functions only 231 are found to be suitable for baseline demodulation. An ideal set of functions should be capable of handling 45 antennas (990 baselines) with that many steps. The problem becomes worse when each antenna is fitted with dual polarization receivers. Since all of the cross products of both polarizations must be measured, the problem increases by a factor of four. An ideal set of functions with 1024 steps would demodulate the signals from only 23 antennas (253 baselines) in a dual polarization system.

One of the purposes of switching is to eliminate cross-talk between the antennas occurring from the antenna to the point where the data is digitized. If this purpose is no longer required then things become much simpler. Simple square waves could be used to eliminate DC and to separate the sidebands. One way to eliminate the need for elaborate demodulation schemes is to digitize the signal at the antennas. Once the signal is digitized there is no longer any danger from cross talk. Another possibility is the use of fiber-optic links from the antennas to the main building. An optical signal must, however, be turned into an electrical one, at some point, in order for it to be digitized. It is at this point that cross-talk becomes a danger.

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