Shifted m-sequences as an alternative to Walsh functions for phase switching in radio interferometry

Eric Keto

Smithsonian Astrophysical Observatory, 60 Garden Street,Cambridge, MA 02138

Submillimeter Array Tech. Memo #134

April 1, 1999

ABSTRACT

Mathematical sequences known as m-sequences can replace Walsh functions in a novel method of phase switching and sideband separation in radio interferometry. The use of shifted m-sequences of four characters allows both the 90° phase changes for sideband separation and the 180° phase changes for phase switching to take place at the first local oscillator and in a shorter switch cycle than by using Walsh functions. The switching sequences based on the shifted m-sequences are orthogonal and provide good cross-talk rejection.

Subject headings: instrumentation

1. Introduction

Phase switching in radio interferometry refers to the technique of periodically reversing the phase of the signal from one antenna and simultaneously reversing the sign of the correlation of the signals from two antennas. Originally introduced by Ryle (1952) as a means of multiplying the voltages from the two antennas, the technique is still used in modern interferometers to reduce noise and DC drifts. Sideband separation refers to the technique of periodically introducing a 90° phase shift into the first local oscillator (LO) at the receivers to form both in-phase and quadrature signals. These signals may be later combined to separate the signals from the upper and lower sidebands of the first down-conversion.

While it is easy to see how to phase switch a two-element interferometer, hold one phase constant and switch the other, the switching patterns become more interesting when more than two antennas are involved. For example, it is not sufficient to switch a third antenna with either pattern of the first two (constant or periodic) if we want to phase switch all three possible correlations. The requirement of sideband separation adds further complexity. Modern interferometers have employed Walsh functions as switching patterns for both phase switching and sideband separation (Wright *et al.* 1973; Granlund, Thompson, and Clark 1978). The Walsh functions are orthogonal binary functions, analogous to the better known sin and cosine orthogonal functions (Harmuth 1970).

In reviewing options for the Submillimeter Array (SMA), we find that the existing switching methods require either a long time for each switch cycle or require the phase switching to be done at the second LO. The sideband separation is always done at the first LO. We would like a short switching cycle because the switch cycle is the minimum averaging time for data and therefore the shortest time scale on which the phase of each antenna may be corrected. For example, it is possible to imagine that a water vapor radiometer operating in real time at each antenna could be used to correct phase on a timescale of a fraction of a second. We would also like to do the phase switching at the first LO because noise and DC drifts will be reduced by the phase switching only if they arise in the system after the phase switch. Thus it is advantageous to have the phase switch as far upstream in the electronics as possible. At the SMA, the first LO is in the receiver in the antenna, but the second LO is in the control building which is over 0.5 km away from the furthest antenna.

Switching the first LO following an m-sequence of four characters, which become the four phases (0°, 90°, 180°, 270°), allows both the phase switching and the sideband separation to be accomplished at the first LO and in a shorter time scale than by using nested Walsh functions at the first LO. The switch cycle time is comparable to that of the method which switches both the first and second LO's using Walsh functions. The m-sequences, short for maximal-length shift register sequences, are integer sequences of three or more characters which have properties analagous to those of the better known pseudo-random binary sequences (Zierler 1959). In particular, the complex auto-correlation function of both pseudo-random binary sequences and m-sequences is equal to unity at zero lag and equal to the inverse of the length of the sequence at all other lags. This property implies that a set of m-sequences which are derived by cyclically shifting a single m-sequence are nearly orthogonal to each other. In this paper we show how to make a set of sequences based on m-sequences which are orthogonal and suitable for sideband separation and phase switching in radio interferometry.

2. Phase switching using Walsh functions

In order to illustrate the method of phase switching and sideband separation we describe two techniques currently in use at other radio interferometers both of which use

Walsh functions.

A simple technique, employed for example at the Owens Valley Radio Observatory, uses two sets of Walsh functions, the first for the 90° sideband separation and the second for the 180° phase switching. All switching is done at the first LO. The phase switching cycles are nested within the sideband separation cycles so that within each of the time steps during which the first LO is in either the 0° or 90° phase of the sideband separation cycle, the first LO is switched by 180° through a complete cycle of another Walsh function. For example, if we use Walsh functions with a length of 8 steps, then a complete cycle of both the 90° and 180° switching will involve 64 steps.

With this switching arrangement, the data may be processed as follows. The correlation of the signals from two antennas will contain a factor of the cosine of the difference in phases between the LO's of two antennas. For example,

$$cos[\Delta\psi_{ij} + \omega_{IF}\tau + \theta_i - \theta_j] \quad (1)$$

where $\Delta\psi_{ij}$ is the phase difference of the signals into antennas i and j (this is the signal to be measured), ω_{IF} is the IF or intermediate frequency after down-conversion, τ is the time lag of the correlation, and θ_i and θ_j are the phases of the LO's of the two antennas.

We first resolve the 180° phase switching cycle which is the inner nest of the two cycles. The phase difference due to this cycle will be either 0° or 180°, and in computing the correlation, we sum the signals with a positive or negative sign according to the sign of the cosine factor. With the 180° phase switching resolved we are left with two streams of data representing the two states arising from the 90° switching. Because $cos(a + \pi/2) = sin(a)$ we have two data streams which represent the in-phase and quadrature components of the measured signal.

Taking the Fourier transform of both these data streams results in,

$$FT[C_{ij}(\tau)] = U_{ij}(\nu) exp(-i\Delta\psi^{u}_{ij}) + L_{ij}(\nu) exp(i\Delta\psi^{\ell}_{ij}) \tag{2}$$

$$FT[S_{ij}(\tau)] = U_{ij}(\nu) i\, exp(-i\Delta\psi^{u}_{ij}) - L_{ij}(\nu) i\, exp(i\Delta\psi^{\ell}_{ij}) \tag{3}$$

where $C_{ij}(\tau)$ is the in-phase correlation, $S_{ij}(\tau)$ is the quadrature correlation, $U_{ij}(\nu)$ and $L_{ij}(\nu)$ are the products of the amplitudes of the upper and lower sideband signals from antennas i and j, and the terms $\Delta\psi_{ij}$ are the phase differences of the incoming signals with the superscripts u and ℓ indicating the sideband. The equation for $FT[S_{ij}(\tau)]$ can be recognized through Euler's identity as a $\pi/2$ shift of the equation for $FT[C_{ij}(\tau)]$. In these equations, the negative frequencies contain no independent information since the measured correlations are purely real functions and the Fourier transform of a purely real function is symmetric. Since we use only the positive frequencies, multiplying $FT[S_{ij}(\tau)]$ by i is equivalent to taking the Hilbert transform of $S_{ij}(\tau)$ followed by the Fourier transform. The sidebands may now be separated as,

$$U_{ij}(\nu) exp(-i\Delta\psi^{u}_{ij}(\nu)) = \frac{1}{2}\Big(FT[C_{ij}(\tau)] - iFT[S_{ij}(\tau)]\Big) \tag{4}$$

$$L_{ij}(\nu) exp(i\Delta\psi^{\ell}_{ij}(\nu)) = \frac{1}{2}\Big(FT[C_{ij}(\tau)] + iFT[S_{ij}(\tau)]\Big) \tag{5}$$

The left hand sides of these last two equations are the desired upper and lower sideband visibility amplitudes and phases for the correlation of antennas i and j.

Walsh functions can be used in a different method of sideband separation and phase switching which does not require nesting. In this method, employed at the Hat Creek Observatory, the in-phase and quadrature signals are generated by switching the first LO by 90° while the 180° signals for phase switching are generated by the second LO. This method is well described in the previous publications, Urry, Thornton, and Hudson (1985) and Welch *et al.* (1996). In this method, the first and second LO's are switched simultaneously. Two independent Walsh functions are required for each receiver to be correlated.

3. Switching with Shifted m-Sequences

The above two examples show that the binary Walsh functions, although serviceable, are not entirely suited to the task of switching the first LO's through the four required phases. It is reasonable to ask whether there are mathematical sequences of four characters which might better serve this role. The 4 character m-sequences referred to in the introduction can be used to generate such a set of orthogonal sequences.

An m-sequence is obtained from a recurrence relation,

$$a_{i+m} = -h_{m-1}a_{i+m-1} - h_{m-2}a_{i+m-2} - \cdots - h_1 a_{i+1} - h_0 a_i \tag{6}$$

where

$$h(x) = x^m + h_{m-1}x^{m-1} + \cdots + h_1 x + h_0 \tag{7}$$

is a primitive polynomial, $h \neq 0$, and h and a are elements of a Galois field of q elements denoted $GF(q)$. The recurrence relation will generate an infinite sequence of period $q^m - 1$ of which any segment of length $q^m - 1$ is an m-sequence. For example, take the case of the Galois field of four elements, $GF(4)$ with the primitive polynomial $h(x) = x^2 + x + w$ of degree $m = 2$. The four elements $0, 1, \omega, \omega^2$ satisfy $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$ so ω is a cube root of unity (Balza, Fromageot, and Maniere 1967; Briggs and Godfrey 1963). We obtain the m-sequence

$$0\ 1\ 1\ \omega^2\ 1\ 0\ \omega\ \omega\ 1\ \omega\ 0\ \omega^2\ \omega^2\ \omega\ \omega^2 \tag{8}$$

For reference, we list an m-sequence of length 63 be generated from the primitive polynomial $x^3 + x^2 + x + \omega$.

$$\begin{gathered} 0\ 0\ 1\ 1\ 0\ \omega^2\ 1\ \omega\ \omega\ \omega\ \omega^2\ \omega\ \omega\ 1\ 0\ \omega\ 0\ \omega\ 1\ \omega^2\ 1 \\ 0\ 0\ \omega\ \omega\ 0\ 1\ \omega\ \omega^2\ \omega^2\ \omega^2\ 1\ \omega^2\ \omega^2\ \omega\ 0\ \omega^2\ 0\ \omega^2\ \omega\ 1\ \omega \\ 0\ 0\ \omega^2\ \omega^2\ 0\ \omega\ \omega^2\ 1\ 1\ 1\ \omega\ 1\ 1\ \omega^2\ 0\ 1\ 0\ 1\ \omega^2\ \omega\ \omega^2 \end{gathered} \tag{9}$$

As with the binary pseudo-random sequences, the autocorrelation function of the m-sequences is the autocorrelation of the complex sequence obtained by replacing each of the elements in an m-sequence by $s_j = exp[2\pi ir/q]$ where r is the number of the element in $GF(q)$.

$$c(j) = \frac{1}{n}\sum_{i=0}^{n-1} s_i \overline{s_{i+j}} \tag{10}$$

$$c(0) = 1$$

$$c(j) = -\frac{1}{q^m - 1}$$

The following combination of properties relevant to our application is equivalent to the autocorrelation property.

i. In an m-sequence of length $q^m - 1$, each non-zero element occurs q^{m-1} times and the element 0 occurs $q^{m-1} - 1$ times.

ii. In the terms of the autocorrelation function, the lags or shifts which are not multiples of $(q^{m-1} - 1)/(q - 1)$ contain each pair of elements q^{m-2} times except the pair $\{0, 0\}$ which occurs $q^{m-2} - 1$ times.

iii. For shifts of $k(q^{m-1} - 1)/(q - 1)$ there are q^{m-1} pairs of the non-zero elements $\{\alpha, \beta^j\alpha\}$, where β is a primitive of the Galois field, and while the pair $\{0, 0\}$ occurs $q^{m-1} - 1$. In the case of $GF(4)$ the last part of this statement means that for the sequence of length 15, the pairs $\{1, \omega\}$, $\{1, \omega^2\}$, $\{\omega, \omega^2\}$ each occur 4 times while the pair $\{0, 0\}$ occurs 3 times.

These properties suggest that the set of $4^m - 1$ cyclic shifts of an m-sequence together with the sequence of zeroes, $\mathbf{0}$, may be used as the switching functions in our application to interferometry.

To use the shifted m-sequences in interferometry, we switch the first LO of each receiver in the pattern given by the cyclically shifted m-sequences over $GF(4)$. For example, in the case of $m = 2$ which generates an m-sequence of length 15, we have 15 cyclic shifts which together with the **0**-sequence provide switching functions for 16 receivers. We need to add to each m-sequence one state in which all the receivers have the same phase. For example we could preface each sequence with one state in which all the receivers have a phase shift of zero. This makes each sequence 16 states long and each switching function now has 4 occurrences of each of the 4 states. The addition of this zero improves the correlation property since as can be seen from equation 10, the missing zero in equation 8 is responsible for the non-zero autocorrelation function at $j \neq 0$. Remember that in the case of the improved m-sequences, we cyclically shift the elements of equation 8, but the balancing zero is always added to the beginning of each shifted sequence of 15 elements.

Since for most combinations of shifted sequences, every pair of elements occurs once, we will have an equal number of phase differences when cross-correlating the antennas. In the case of pairs of sequences where the shift is a multiple of $(q^{m-1} - 1)/(q - 1)$ (for length 15 these are the shifts of 5 and 10), we are missing one of the 4 phase differences. To compensate, we need to double the sequence length and in the second half of the sequence, reverse the phases of either the $0 - 180°$ pair or the $90 - 270°$ pair. Alternatively we could use a longer m-sequence, say m=3, of length 63. In the case of 16 receivers we can use the first 15 cyclic shifts since the problem of the missing phases does not occur until a shift by $(q^{m-1} - 1)/(q - 1) = 21$ elements.

To perform the sideband separation, we follow the same algorithm we used with the nested Walsh functions. At the end of each switch cycle we first resolve the phase differences of 180° obtaining the two data sets, equations 2 and 3 with a phase difference of 90° degrees.

Let us take the case of $q = 4$, $m = 2$, write down some cycles and verify the properties. The difference between obtaining and verifying a switching function is similar to the difference between obtaining and verifying a solution to a differential equation. The verification is straightforward. We will use the integers $1, 2, 3, 4$ to represent the four phase shifts of $0, 90, 180, 270$ degrees. Antenna 0 will be assigned the switch function based on the sequence in equation 8 and antenna 1 will be based on equation 8 shifted by 1. Remember that we do not shift the whole sequence for antenna 0, but we cyclically shift the first half and second half separately, and we do not shift the zeroes prefacing each half.

$$Ant\ 0: \ 00113102212033230033130223201121 \tag{11}$$

$$Ant\ 1: \ 03011310221203320103313022320112 \tag{12}$$

The phase differences of each state are obtained by integer arithmetic mod 4.

$$Diff\ 0-1: \ 01102232031230310330221201321013 \tag{13}$$

Now let us verify the properties for antenna 5. The difference $5-0$ shows how the pattern works when the shift is by a multiple of $(q^{m-1}-1)/(q-1) = 5$

$$Ant\ 5: \ 00332301131022120011210331302232 \tag{14}$$

$$Diff\ 5-0: \ 00223203323033330022120112101111 \tag{15}$$

$$Diff\ 1-5: \ 03123013130221200132103131022320 \tag{16}$$

The 16 sequences generated this way are mutually orthogonal in that

$$c_{ij} = \frac{1}{n} \sum_{k=0}^{n-1} s_k^i \overline{s_k^j} \tag{17}$$

$$c_{ij} = 1 \quad i = j$$

$$c_{ij} = 0 \quad i \neq j$$

where s_k^i is the kth element of the ith complex sequence formed as in equation 10, but here from the switching patterns rather than the original m-sequences. The switching sequences above also have a higher phase switching frequency than the low order Walsh functions. These properties make for excellent cross-talk rejection.

4. Application of m-sequences to the SMA

In the case of the SMA, we have 16 receivers that can be cross-correlated and the minimum sample time is 10 ms. To use the method of nested Walsh functions requires 16 independent Walsh functions of length 16 steps (the same set of 16 Walsh functions can be used for both the phase switching and sideband separation cycles). If each step contains the minimum 10 ms of data, one complete cycle requires 2.56 seconds. Using the second method in which both the first and second LO's are switched, the SMA with 16 receivers would require 32 Walsh functions resulting in a cycle time of 0.32 s. While this cycle time is acceptably short, the disadvantage for the SMA is that the second LO's of the SMA are in the control building rather than the antennas. Drifts and spurious signals generated in the antennas and transmission lines before the second LO's will not be removed by this phase switching method.

Using the shifted m-sequences, we can perform both the phase switching and the switching for sideband separation at the first LO and in a shorter cycle period than by using nested Walsh functions. The shortest cycle period for 16 or fewer cross-correlated receivers is 32 steps or 0.32 s (with a sample time of 10 ms) as opposed to 2.56 seconds with the nested Walsh functions. The patterns based on m-sequences are orthogonal and also have a higher switching frequency than those based on low order Walsh functions.

REFERENCES

Balza, C., Fromageot, A., and Maniere, M., 1967, Four-level pseudo-random sequences, Electron. Lett. 3, 313-315

Briggs, P.A.N. and Godfrey, K.R., 1963, Autocorrelation function of a 4-level m-sequence, Electon. Lett., 4, 232-233

Granlund, J., Thompson, A.R., Clark, B.G., 1978 An Application of Walsh Functions in Radio Astronomy Instrumentation, IEEE Trans. Electromagn. Comput. EMC-20, 451-453

Harmuth, H.F, 1970, Transmission of Information by Orthogonal Functions, Springer-Verlag, Berlin

Ryle, M.A., 1952, A New Radio Interferometer and it Application to the Observation of Weak Radio Stars, Proc. R. Soc. A, 211, 351-375

Welch, W.J., et al., 1996, The Berkeley-Illinois-Maryland-Association Millimeter Array, PASP, 108, 93-103

Wright, M.C.H., Clark, B.G., Moore, C.H., and Coe, J., 1973, Hydrogen-line aperture synthesis at the National Radio Astronomy Observatory: Techniques and data reduction, Radio Science, 8, 763

Zierler, N, 1959, Linear Recurring Sequences, J.Soc. Indust. Appl. Math., 7, 31-48

This manuscript was prepared with the AAS LaTeX macros v4.0.